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Love waves in layered anisotropic media ☆

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Abstract

A modified transfer-matrix method is proposed to describe Love waves in multilayered anisotropic (monoclinic) media. Dispersion relations for media consisting of one and two anisotropic elastic layers in contact with an anisotropic half-space are obtained in closed form. The conditions for Love waves to exist are analysed. Waves with horizontal transverse polarization of the non-canonical type are investigated.

1. Introduction

Love waves in an isotropic layer in contact with an isotropic half-space. In addition to Rayleigh waves, Love waves[1] play an important role in the transmission of seismic of energy and are very often recorded during seismic activity and explosions. The displacement field corresponding to a Love wave can be represented in the form

$$\mathbf{u}_1(\mathbf{x}) = \mathbf{m}(C_1 e^{-ir\gamma_1 x'} + C_2 e^{ir\gamma_2 x'})e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)}, \quad \mathbf{u}_2(\mathbf{x}) = \mathbf{m}(C_3 e^{ir\gamma_2 x'})e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{1.1}$$

where $\mathbf{u}_1$ is the displacement in the layer, $\mathbf{u}_2$ is the displacement in the half-space and $\mathbf{m}$ is the unit amplitude (the polarization vector). It is assumed that the vector $\mathbf{m}$ is orthogonal to the sagittal plane (it is defined by the vector $\mathbf{n}$, which specifies the direction of propagation of the wave front and by the unit vector $\boldsymbol{\nu}$ normal to the free surface); $x' \equiv \boldsymbol{\nu}\cdot\mathbf{x}$ is the coordinate along the vector $\boldsymbol{\nu}$, and it is henceforth assumed that x' takes negative values in the half-space, r is the wave number, c is the phase velocity and t is the time; the unknown (complex) coefficients C_k are found, apart from a factor, from the boundary conditions on the external plane boundary

$$x' = 2h: \mathbf{t}_\mathbf{v} \equiv \mathbf{v}\cdot\mathbf{C}_1 \cdot\cdot \nabla_\mathbf{x}\mathbf{u}_1 = 0 \tag{1.2}$$

and the contact conditions at the interface

$$x' = 0: \mathbf{v}\cdot\mathbf{C}_1 \cdot\cdot \nabla_\mathbf{x}\mathbf{u}_1 = \mathbf{v}\cdot\mathbf{C}_2 \cdot\cdot \nabla_\mathbf{x}\mathbf{u}_2, \quad \mathbf{u}_1 = \mathbf{u}_2 \tag{1.3}$$

The parameters γ_1 and γ_2 correspond to the complex roots of Christoffel's equation (it will be derived later), $\mathbf{C}_1$ and $\mathbf{C}_2$ are four-valent elasticity tensors of the layer and half-space, respectively, and 2 h is the layer thickness.

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Remark 1.1. Corresponding to representation (1.1), attenuation in the depth of the half-space is ensured by the Christoffel parameter γ_2 with negative imaginary part.

The following assertion is attributed to Love.

Proposition 1.1.

A. *Surface waves with transverse horizontal polarization may occur in an isotropic layer and in an isotropic half-space in contact with it only when the phase velocity satisfies the condition*

$$c_1^T < c < c_2^T; \quad c_k^T = \sqrt{\mu_k/\rho_k}, \quad k = 1, 2 \tag{1.4}$$

where c_k^T are the velocities of transverse bulk waves in the layer ($k=1$) and in the half-space ($k=2$), μ_k are the corresponding Lamé constants and ρ_k are densities.

B. *The dispersion relation between the phase velocity c and the frequency ω can be represented in the form*

$$\omega = \frac{c}{2h\sqrt{g_1-1}}\left(\operatorname{arctg}\left(\frac{\mu_2}{\mu_1}\sqrt{\frac{1-g_2}{g_1-1}}\right)+n\pi\right), \quad n = 0, 1, 2, \ldots; \quad g_k = \frac{\rho_k c^2}{\mu_k}, \quad k = 1, 2 \tag{1.5}$$

Corollary 1.1.

A. *For a fixed frequency ω a finite number of Love waves exist, which propagate with different phase velocities $c \in (c_1^T, c_2^T)$.*

B. *For a fixed phase velocity $c \in (c_1^T, c_2^T)$ an infinite number of Love waves exist with different frequencies ω.*

Corollary 1.2. *No Love waves exist if $c_1^T > c_2^T$.*

Love waves in an anisotropic layer in contact with an anisotropic half-space. It was shown in Ref. 2 that Love waves can propagate in both an isotropic medium and in a system consisting of an anisotropic layer and in an anisotropic half-space in contact with it, in which case it was assumed that the layer and the half-space have an axis of elastic symmetry of the fourth or sixth order, oriented along the polarization vector **m**. For such a system, the propagation conditions and the dispersion relations are similar to the isotropic case. Using the approach developed in Ref. 2, dispersion relations were obtained in Ref. 3 for a transversely isotropic layer and for a transversely isotropic half-space in contact with it.

Love waves in multilayered media. For multilayered media, consisting of two or more layers, in contact with a half-space, analytical solutions exist similar to the solution for a single-layer medium. The dispersion relations for a Love wave in multilayered media can be obtained numerically using two matrix methods, initially proposed for analysing Love waves. These approaches are known as the transfer-matrix method (sometimes called the Thomson-Haskell method, after the developers[4,5]) and the global-matrix method.[6,7] The transfer-matrix method is based on the successive solution of contact boundary-value problems at the interfaces and the construction of corresponding transfer matrices. We will discuss this method in more detail below. The global-matrix method is based on the solution of ordinary differential equations with piecewise-uniform coefficients, which lead, in the final analysis, to the construction of a special "global" matrix.

When the global matrix method appeared it was assumed that, in numerical realizations, it leads to (numerically) more stable solutions than the transfer-matrix method. Later various modifications of the two methods were proposed in order to make them more numerically stable (see Ref. 8–14). The problem of numerical stability becomes particularly urgent when there is a large number of layers. In this case the transfer-matrix method begins to exhibit advantages, since the order of the corresponding matrices remains unchanged when the number of layers changes (in the case of the global-matrix method the order of the corresponding matrix increases linearly with the number of layers).

Below we develop a modified transfer-matrix method based on the use of hyperbolic functions in the representation for partial waves and intended both for an analytical investigation of Love waves in anisotropic media with from one to three layers, and for a numerical analysis of systems containing a large number of layers. It will be shown later that representation (1.1) turns out to be incorrect if multiple roots occur in Christoffel's equation (there is an analogy here with Rayleigh and Lamb waves, whose structure changes when multiple roots arise, see Ref. 15–17).

A correct representation for Love waves and the corresponding modification of the transfer-matrix method are given below. Moreover, the transfer-matrix method will be used to obtain the resolvents for waves with transverse horizontal polarization, propagating in multilayered plates with free and restrained boundary surfaces.

2. Fundamental relations

Below, both the layer and the half-space will be assumed to be uniform and linearly hyperelastic. The equations of motion for a uniform elastic anisotropic medium can be represented in the form

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \mathrm{div}_x \mathbf{C} \cdot\cdot \nabla_x \mathbf{u} - \rho\ddot{\mathbf{u}} = 0 \tag{2.1}$$

where the four-valent elasticity tensor $\mathbf{C}$ is assumed to be positive-definite

$$(\mathbf{A}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{A}) \equiv \sum_{i,j,m,n} A_{ij}C^{ijmn}A_{mn} > 0, \qquad \underset{\mathbf{A} \in \mathrm{sym}(R^3 \otimes R^3),\ \mathbf{A} \neq 0}{\forall \mathbf{A}}, \qquad \mathrm{sym}\mathbf{X} = \frac{1}{2}(\mathbf{X} + \mathbf{X}^t) \tag{2.2}$$

Remark 2.1.

A. It is assumed that all the media considered have planes of elastic symmetry, coinciding with the sagittal plane $\mathbf{m} \cdot \mathbf{x} = 0$. This is equivalent to the presence in the elasticity tensor of a monoclinic symmetry group. It can be shown,[18] that this implies that all the separable components of the elasticity tensor are equal to zero in which the vector $\mathbf{m}$ appears an odd number of times (in an orthogonal basis in R^3, formed by the vector $\mathbf{m}$ and any orthogonal vectors belonging to the sagittal plane). (Components that can be represented in the form of the tensor product of vectors from R^3 are said to be separable.) In the case of monoclinic symmetry the elasticity tensor contains 13 independent separable components.

B. It will be shown below that monoclinic symmetry is a sufficient condition for collinearity of the vector $\mathbf{m}$ of the surface forces in any plane $\boldsymbol{\nu} \cdot \mathbf{x} = \mathrm{const}$.

Following the method described earlier in Ref. 15,16, we will consider the following representation for the Love wave, which is more general than that in (1.1)

$$\mathbf{u}(\mathbf{x}) \equiv \mathbf{m} f(irx')e^{ir(\mathbf{n} \cdot \mathbf{x} - ct)} \tag{2.3}$$

where $x' = \boldsymbol{\nu} \cdot \mathbf{x}$, as in representation (1.1), f is an unknown scalar function, the exponential factor on the right-hand side of (2.3) corresponds to the propagation of the wave front in the direction $\mathbf{n}$ with phase velocity c, and r is the wave number.

Substituting expression (2.3) into Eq. (2.1) and bearing Remark 2.1. A in mind, we obtain a differential equation, the characteristic equation of which, known as Christoffel's equation, has the form

$$\begin{gathered}(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma^2 + \\ + (\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma + (\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m} - \rho c^2) = 0\end{gathered} \tag{2.4}$$

The left-hand side of Eq. (2.4) is a second-order polynomial in the Christoffel parameter γ. Hence, for the elastic symmetry investigated, the Love wave in the layer can consist of only two partial waves.

3. The displacements and surface forces in the half-space

We shall henceforth assume that the multilayered medium consists of n layers, which, unless otherwise stated, are in contact with the half-space, and the subscript $n + 1$ will relate to the half-space. Since the displacement field in the half-space should attenuate with depth, the corresponding root of Christoffel's equation should be complex with a negative imaginary part (see Remark 1.1).

Proposition 3.1. *Attenuation with depth in a monoclinic half-space is possible if and only if the phase velocity c lies in the following range:*

$$c \in (0, [\rho_{n+1}^{-1}(E_{n+1} - F_{n+1}^2/G_{n+1})]^{1/2}) \tag{3.1}$$

$$E_{n+1} = \mathbf{m} \otimes \mathbf{n} \cdot\cdot\, \mathbf{C}_{n+1} \cdot\cdot\, \mathbf{n} \otimes \mathbf{m}, \quad F_{n+1} = \mathbf{m} \cdot \mathrm{sym}(\mathbf{n} \cdot \mathbf{C}_{n+1} \cdot \boldsymbol{\nu}) \cdot \mathbf{m}$$

$$G_{n+1} = \mathbf{m} \otimes \boldsymbol{\nu} \cdot\cdot\, \mathbf{C}_{n+1} \cdot\cdot\, \boldsymbol{\nu} \otimes \mathbf{m}$$

The corresponding Christoffel parameter

$$\gamma_{n+1} = -F_{n+1}/G_{n+1} - i((E_{n+1} - \rho_{n+1}c^2)/G_{n+1} - (F_{n+1}/G_{n+1})^2)^{1/2} \tag{3.2}$$

has a negative imaginary part.

Proof. A direct analysis of the roots of Christoffel's equation (2.4) shows that they are complex with a negative discriminant, which gives an upper boundary of the range (3.1). The fact that the radicand in (3.1) is positive follows from an analysis of the quadratic polynomial

$$P(x) \equiv \mathbf{m} \otimes (x\boldsymbol{\nu} + \mathbf{n}) \cdot\cdot\, \mathbf{C}_{n+1} \cdot\cdot\, (\mathbf{n} + x\boldsymbol{\nu}) \otimes \mathbf{m} \tag{3.3}$$

It is positive for any real x, since the elasticity tensor is positive definite. The non-existence of real roots of this polynomial completes the proof, since the discriminant of this polynomial is identical (apart from the factor $-(\rho_{n+1})^{-1/2}$) with the upper limit of the range (3.1). It only remains to point out that expression (3.2) is obtained from the solution of Christoffel's equation. □

Corollary 3.1. *The parameter* γ_{n+1} cannot be a multiple root of Christoffel's equation.

The proof follows from the condition for the discriminant of Eq. (3.2) *to be non-zero, which is necessary for the solution to attenuate with depth.*

Corollary 3.2. *If the material considered again possesses a single plane of elastic symmetry, the normal to which coincides with the vector* $\boldsymbol{n}$ *or* $\boldsymbol{\nu}$ *(such a material is necessarily orthotropic), the permissible velocity range has the form*

$$c \in (0, c_{n+1}^T) \tag{3.4}$$

where c_{n+1}^T *is the velocity of the transverse body (seismic) wave, propagating in the direction of the vector* $\boldsymbol{n}$ *and polarized in the direction of the vector* $\boldsymbol{m}$*. For the case considered, the Christoffel parameter is pure imaginary*

$$\gamma_{n+1} = -i((E_{n+1} - \rho_{n+1}c^2)/G_{n+1})^{1/2} \tag{3.5}$$

Proof. For such a material the quantity F_{n+1} in expression (3.2) becomes equal to zero, since the vectors $\mathbf{n}$ and $\boldsymbol{\nu}$ occur in it an odd number of times. It is further sufficient to note that when $F_{n+1} = 0$ the radicand in (3.1) is identical with the expression for c_{n+1}^T. The remaining part of the proof follows from an analysis of Christoffel's equation.

Representation (1.1) for the half-space gives the following field of surface forces in the $\boldsymbol{\nu} \cdot \mathbf{x} = 0$ plane:

$$\begin{aligned} \mathbf{t}_{n+1}(\mathbf{x})|_{\boldsymbol{\nu} \cdot \mathbf{x} = 0} &= irC_{2n+1}(\gamma_{n+1}\mathbf{a}_{n+1} + \mathbf{b}_{n+1})e^{ir(\mathbf{n} \cdot \mathbf{x} - ct)} \\ \mathbf{a}_{n+1} &= \boldsymbol{\nu} \cdot \mathbf{C}_{n+1} \cdot\cdot\, \boldsymbol{\nu} \otimes \mathbf{m}, \quad \mathbf{b}_{n+1} = \boldsymbol{\nu} \cdot \mathbf{C}_{n+1} \cdot\cdot\, \mathbf{n} \otimes \mathbf{m} \quad \Box \end{aligned} \tag{3.6}$$

Proposition 3.2. *The surface forces* (3.6) *are collinear with the vector* $\boldsymbol{m}$*.*

The proof follows from the assumption that all the materials belong to the monoclinic system (it has a symmetry group generated by rotations of $\mathbf{R}_{\mathbf{m}}^{\pi}$), which ensures that the vector $\mathbf{m}$ has an even form in the separable components of the tensor $\mathbf{C}_{n+1}$ in the basis formed by the vectors $\mathbf{m}$, $\boldsymbol{\nu}$ and $\mathbf{n}$.

4. Displacements and surface forces in layers

In this section the subscript $k(1 \le k \le n)$ relates to the corresponding layer.

4.1. Simple (non-multiple) roots

For simple roots of Christoffel equation and an orthotropic material with principal axes of elasticity coinciding with the vectors **m**, **n** and $\boldsymbol{\nu}$, representation (1.1) remains true. Nevertheless, for the purposes of the present analysis, which includes a more general class of materials, belonging to the monoclinic system, representation (1.1) will be modified as follows:

$$\mathbf{u}_k(\mathbf{x}) = \mathbf{m}(C_{2k-1}\upsilon_k^{(1)} + C_{2k}\upsilon_k^{(2)})e^{ir(\beta_k x' + \mathbf{n}\cdot\mathbf{x} - ct)} \quad (\gamma_k = \alpha_k + \beta_k) \tag{4.1}$$

where

$$\upsilon_k^{(1)} = \operatorname{sh}(ir\alpha_k x'), \quad \upsilon_k^{(2)} = \operatorname{ch}(ir\alpha_k x')$$
$$\alpha_k = -((E_k - \rho_k c^2)/G_k - (F_k/G_k)^2)^{1/2}, \quad \beta_k = -F_k/G_k \tag{4.2}$$

The parameters E_k, F_k and G_k are defined by formulae (3.1). It can be seen that α_k is a real or imaginary parameter depending on the value of the phase velocity, while β_k is a real parameter independent of c.

Bearing representation (4.1) in mind, we will write the corresponding surface forces in the $\boldsymbol{\nu}\cdot\mathbf{x} = x'$ plane in the form

$$\mathbf{t}_k(x') = i[(\mathbf{a}_k(\alpha_k\upsilon_k^{(2)} + \beta_k\upsilon_k^{(1)}) + \mathbf{b}_k\upsilon_k^{(1)})C_{2k-1} +$$
$$+ (\mathbf{a}_k(\alpha_k\upsilon_k^{(1)} + \beta_k\upsilon_k^{(2)}) + \mathbf{b}_k\upsilon_k^{(2)})C_{2k}]re^{ir(\beta_k x' + \mathbf{n}\cdot\mathbf{x} - ct)} \tag{4.3}$$

Proposition 4.1. *The surface forces (4.3) are collinear with the vector* ***m****.*

The proof is similar to the proof of Proposition 3.2.

Bearing representation (4.3) in mind and also the fact that $\mathbf{b}_k = 0$ and $\gamma_k = \alpha_k$ ($\beta_k = 0$) for an orthotropic material with axes of elastic symmetry coinciding with the vectors **m**, **n** and $\boldsymbol{\nu}$, we obtain the following expression for the surface forces

$$\mathbf{t}_k(x') = ir\gamma_k\mathbf{a}_k(C_{2k-1}\upsilon_k^{(2)} + C_{2k}\upsilon_k^{(1)})e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)}, \quad \alpha_k = \gamma_k \tag{4.4}$$

4.2. Multiple roots

Representation (4.1) for the Love waves in the layer become incorrect when multiple roots occur in Christoffel's equation. This happens when the parameter α_k, defined by the third relation of (4.2), vanishes.

Proposition 4.2.

A. *The phase velocity for which multiple roots occur is described by the expression*

$$c = (\rho^{-1}(E_k - F_k^2/G_k))^{1/2} \tag{4.5}$$

B. *The corresponding Christoffel parameter (necessarily real) has the form*

$$\gamma_k = -F_k/G_k \tag{4.6}$$

C. *The representations of the displacement field and of the corresponding surface forces in the* $\boldsymbol{\nu}\cdot\mathbf{x} = x'$ *plane, corresponding to multiple roots, have the form*

$$\mathbf{u}_k(\mathbf{x}) = \mathbf{m}(C_{2k-1} + iex'C_{2k})e^{ir(\gamma_k x' + \mathbf{n}\cdot\mathbf{x} - ct)} \tag{4.7}$$

$$\mathbf{t}_k(x') = i(\mathbf{a}_k(\gamma_k C_{2k-1} + (1 + ir\gamma_k x')C_{2k}) + \mathbf{b}_k(C_{2k-1} + irx'C_{2k}))re^{ir(\gamma_k x' + \mathbf{n}\cdot\mathbf{x} - ct)} \tag{4.8}$$

Proof. Parts of assertions A and B follow from the condition for the discriminant on the right-hand side of Eq. (3.2) to vanish. Representations C correspond to the general solution of Eq. (2.4) in the case of multiple roots (see also Ref. 15–17). □

Proposition 4.3. *The surface forces (4.8) are collinear with the vector* $\boldsymbol{m}$.

The proof is similar to the proof of Proposition 3.2.

5. The modified transfer-matrices method

5.1. The transfer matrices

In accordance with Propositions 4.1 and 4.2, the scalar amplitudes of the displacements and of the surface forces in the k-th layer in the $\boldsymbol{\nu}\cdot\mathbf{x}=x'$ plane can be represented in the form

$$\left\|\begin{matrix} u_k(x') \\ t_k(x') \end{matrix}\right\| = \mathbf{M}_k(x')\left\|\begin{matrix} C_{2k-1} \\ C_{2k} \end{matrix}\right\|;\ u_k(x') \equiv \left|\mathbf{u}_k(x')e^{-ir(\mathbf{n}\cdot\mathbf{x}-ct)}\right|,\ t_k(x') \equiv \left|\mathbf{t}_k(x')e^{-ir(\mathbf{n}\cdot\mathbf{x}-ct)}\right| \tag{5.1}$$

where $u_k(x')$ and $t_k(x')$ are the corresponding scalar amplitudes and $\mathbf{M}_k$ is a 2×2 matrix.

Bearing expressions (4.1)–(4.3), (4.5) and (4.6) in mind, and introducing the notation $H_k=\mathbf{m}\cdot\mathbf{b}_k$, we will write the elements of the matrix $\mathbf{M}_k$ in the following form:

in the case of simple roots

$$M_{1l(k)} = \upsilon_k^{(l)}e^{ir\beta_k x'},\quad M_{2l(k)} = i(G_k(\alpha_k\upsilon_k^{(3-l)}+\beta_k\upsilon_k^{(l)})+H_k\upsilon_k^{(l)})re^{ir\beta_k x'};\quad l = 1, 2 \tag{5.2}$$

in the case of multiple roots

$$\begin{aligned} &M_{11(k)} = e^{ir\gamma_k x'},\quad M_{12(k)} = irx'e^{ir\gamma_k x'} \\ &M_{21(k)} = i(\gamma_k G_k + H_k)re^{ir\gamma_k x'},\quad M_{22(k)} = i((1+ir\gamma_k x')G_k + irx'H_k)re^{ir\gamma_k x'} \end{aligned} \tag{5.3}$$

Note that, in accordance with the last expression of (4.2), the parameter β_k in relations (5.2) is independent of the phase velocity c.

Proposition 5.1. *Irrespective of the dependence on the multiplicity of the roots, the matrices* $\mathbf{M}_k$ *are not degenerate for any real* x'.

Proof. Note that the exponential factors in relations (5.2) and (5.3) are nonzero for any x'; a further direct analysis shows that the matrices with elements (5.2) and (5.3) are not degenerate for any x' (the determinant of the matrix with elements (5.2) is equal to $-\alpha_k G_k e^{ir\beta_k x'}$ with $\alpha_k \neq 0$, since the roots are simple; the determinant of the matrix with elements (5.3) is equal to $G_k e^{ir\gamma_k x'}$).

Now, using the matrices $\mathbf{M}_k$, the displacements and surface forces at the interface between the n-th layer and the half-space can be represented solely in terms of the coefficients C_1 and C_2

$$\left\|\begin{matrix} u_n(-h_n) \\ t_n(-h_n) \end{matrix}\right\| = \left(\prod_{k=2}^{n}(\mathbf{M}_k(-h_k)\cdot\mathbf{M}_k^{-1}(h_k))\right)\cdot\mathbf{M}_1(-h_1)\left\|\begin{matrix} C_1 \\ C_2 \end{matrix}\right\| \tag{5.4}$$

where $2h_k (k=1, \ldots, n)$ are the thicknesses of the corresponding layers. □

5.2. The boundary conditions on the outer surface

Expressions (4.3) and (4.8) enable us to write the conditions for there to be no shear stresses on the outer surface of the first layer in the form

$$t_1(h_1) \equiv \tilde{\mathbf{B}}_1(h_1) \cdot \tilde{\mathbf{C}} = 0; \quad \tilde{\mathbf{B}}_1(h_1) = (X_1(h_1), Y_1(h_1)), \quad \tilde{\mathbf{C}} = (C_1, C_2) \tag{5.5}$$

where t_1 is the corresponding scalar amplitude, and here and henceforth quantities with a tilde denote two-dimensional vectors.

Introducing the notation

$$w_k^{(1)} = \operatorname{sh}(ir\alpha_1 h_1), \quad w_k^{(2)} = \operatorname{ch}(ir\alpha_1 h_1), \quad H_1 = \mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_1 \cdot\cdot \mathbf{n} \otimes \mathbf{m} \tag{5.6}$$

we obtain:

in the case of simple roots

$$\begin{aligned} X_1(h_1) &= (G_1(\alpha_1 w_1^{(2)} + \beta_1 w_1^{(1)}) + H_1 w_1^{(1)}) ire^{ir\beta_1 h_1}, \\ Y_1(h_1) &= (G_1(\alpha_1 w_1^{(1)} + \beta_1 w_1^{(2)}) + H_1 w_1^{(2)}) ire^{ir\beta_1 h_1} \end{aligned} \tag{5.7}$$

in the case of multiple roots

$$X_1(h_1) = (\gamma_1 G_1 + H_1) ire^{ir\gamma_1 h_1}, \quad Y_1(h_1) = ((1 + ir\gamma_1 h_1) G_1 + irh_1 H_1) ire^{ir\gamma_1 h_1} \tag{5.8}$$

Equations (5.1) enable us to express the coefficients C_1 and C_2 (apart from a factor) in the form of the solution of the following equation

$$\tilde{\mathbf{T}}_1(h_1) \times \tilde{\mathbf{C}} = 0; \quad \tilde{\mathbf{T}}_1(h_1) = (-Y_1(h_1), X_1(h_1)) \tag{5.9}$$

It can be seen that the two-dimensional vectors $\tilde{\mathbf{T}}(h_1)$ and $\tilde{\mathbf{C}}$ are collinear.

5.3. The boundary conditions at the interface of the n-th layer and the half-space

The boundary conditions at the corresponding interface can be represented in the form

$$\tilde{\mathbf{V}}_n(-h_n) \cdot \tilde{\mathbf{W}}_{n+1}(0) = 0 \tag{5.10}$$

where

$$\begin{aligned} &\tilde{\mathbf{V}}_n(-h_n) = (u_n(-h_n), t_n(-h_n)), \quad \tilde{\mathbf{W}}_{n+1}(0) = (-t_{n+1}(0), u_{n+1}(0)); \\ &t_{n+1}(0) = \left|\mathbf{t}_{n+1}(0) e^{-ir(\mathbf{n} \cdot \mathbf{x} - ct)}\right| \end{aligned} \tag{5.11}$$

(the vector $\mathbf{t}_{n+1}(0)$ is defined by formula (3.6)).

Bearing relations (5.11) in mind, it can be seen that Eq. (5.10) expresses the collinearity of the vectors $\tilde{\mathbf{V}}_n$ and $(u_{n+1}(0), t_{n+1}(0))$, the latter being equivalent to the condition for the displacements and forces to be collinear when passing through the interface of the components. When writing the second relations of (5.11) it was assumed that, in the local system of coordinates for the half-space, the interface is specified by the equation $\boldsymbol{\nu} \cdot \mathbf{x} = 0$.

5.4. The resolvent for the Love waves

Bearing relations (5.4), (5.9), (5.10) and the second relation of (5.11) in mind, the combined equation of the modified transfer-matrices method can be represented in the form

$$\tilde{\mathbf{W}}_{n+1}(0) \cdot \left(\left(\prod_{k=2}^{n} \mathbf{M}_k(-h_k) \cdot \mathbf{M}_k^{-1}(h_k) \right) \cdot \mathbf{M}_1(-h_1) \right) \cdot \tilde{\mathbf{T}}(h_1) = 0 \tag{5.12}$$

This is the required resolvent for the Love waves.

5.5. The resolvent for horizontally polarized shear waves in multilayered plates

We will obtain the resolvent for horizontally polarized shear waves in a plate consisting of n layers ($n>1$). The outer surfaces of the plate are assumed to be either free from surface forces or constrained, or mixed boundary conditions are specified on them (one of the surfaces is stress-free while the other is restrained).

1°. For a multilayered plate with free outer surfaces the boundary conditions have the form

$$\mathbf{t}_1(h_1) \equiv \boldsymbol{\nu} \cdot \mathbf{C}_1 \cdot \nabla \mathbf{u} = 0, \quad \mathbf{t}_n(-h_n) \equiv \boldsymbol{\nu} \cdot \mathbf{C}_n \cdot \nabla \mathbf{u} = 0 \tag{5.13}$$

By analogy with the above, the use of the modified transfer-matrices method enables us to obtain the resolvent in the form

$$\tilde{\mathbf{T}}_n(-h_n) \cdot \mathbf{N}_n \cdot \tilde{\mathbf{T}}_1(h_1) = 0, \quad \mathbf{N}_n = \mathbf{M}_n^{-1}(h_n) \cdot \left(\prod_{k=2}^{n-1} \mathbf{M}_k(-h_n) \cdot \mathbf{M}_k^{-1}(h_k) \right) \cdot \mathbf{M}_1(-h_1)$$
$$\tilde{\mathbf{T}}_1 = (-Y_1(h_1), X_1(h_1)), \quad \tilde{\mathbf{T}}_n(-h_n) = (X_n(-h_n), Y_n(-h_n)) \tag{5.14}$$

The components X_k and Y_k ($k=1, n$) are found from formulae (5.7) and (5.8).

2°. For a multilayered plate with restrained outer surfaces, the boundary conditions have the form

$$\mathbf{u}_1(h_1) = 0, \quad \mathbf{u}_n(-h_n) = 0 \tag{5.15}$$

The use of the modified transfer-matrices method gives the following resolvent

$$\tilde{\mathbf{D}}_n(-h_n) \cdot \mathbf{N}_n \cdot \tilde{\mathbf{D}}_1(h_1) = 0; \ \tilde{\mathbf{D}}_1(h_1) = (-U_1(h_1), S_1(h_1)), \ \tilde{\mathbf{D}}_n(-h_n) = S_n(-h_n), U_n(-h_n) \tag{5.16}$$

$S_k(\pm h_k) = \pm w_k^{(1)}$, $U_k(\pm h_k) = w_k^{(2)}$ in the case of simple roots and ($S_k(\pm h_k)=1$, $U_k(\pm h_k)=\pm i r h_k$ in the case of multiple roots. Here we have used relations (4.1) and (4.7) and the notation (5.6), while the parameter α_k is given by the third formula of (4.2).

3°. For a multilayered plate, the upper surface of which is free while the lower is restrained, the boundary conditions have the form

$$\mathbf{t}_1(h_1) = 0, \quad \mathbf{u}_n(-h_n) = 0 \tag{5.17}$$

The resolvent has the form

$$\tilde{\mathbf{D}}_n(-h_n) \cdot \mathbf{N}_n \cdot \tilde{\mathbf{T}}_1(h_1) = 0 \tag{5.18}$$

The modification of this equation for the case when the upper surface is restrained while the lower one is free is obvious.

Remark 5.1. The resolvents (5.12), (5.14), (5.16) and (5.18) can be regarded as implicit equations in the wave number r for a fixed phase frequency ω. Using the relation $r=\omega/c$ in these equations we can obtain a (generally speaking, implicit) dispersion equation, expressing the relation between the phase frequency and the phase velocity.

6. Some analytical solutions

6.1. A single uniform layer in an orthotropic half-space

Suppose the vectors $\boldsymbol{\nu}$, $\mathbf{n}$ and $\mathbf{m}$ coincide with the principal axes of the elasticity of the layer and the half-space. For this case, the Christoffel parameters take the form

$$\gamma_k = (-1)^{k+1} i[(E_k - \rho_k c^2)/G_k]^{1/2}, \quad k = 1, 2 \tag{6.1}$$

Here and henceforth in this section the subscript 1 relates to the layer and the subscript 2 refers to the half-space.

Remark 6.1.

A. Expression (6.1) when $k=1$ shows that multiple roots of the characteristic equation for the layer can arise only if the phase velocity is identical with the velocity of the transverse body wave, which propagates in the direction **n** and is polarised in the direction of the vector **m**.
B. Despite the fact that some analytical results for an orthotropic layer lying on an orthotropic half-space are known,[2] the corresponding explicit dispersion equation has not been obtained.

The scalar amplitude of surface forces, defined by the last formula of (5.1) with $k=1$, in the $\boldsymbol{\nu}\cdot\mathbf{x}=x'$ plane in the layer has the form

$$t_1(x') = ir\gamma_1 G_1(C_1 \operatorname{ch}(ir\gamma_1 x') + C_2 \operatorname{sh}(ir\gamma_1 x')) \text{ in the case of simple roots} \tag{6.2}$$

$$t_1 = irG_1C_2 \text{ in the case of multiple roots} \tag{6.3}$$

The scalar amplitude of the surface forces, defined by the last formula of (5.1) where $k=2$, acting in the $x'=0$ plane in the half-space, has the form

$$t_2(0)|_{\boldsymbol{\nu}\cdot\mathbf{x}=0} = ir\gamma_2 G_2 C_3 \tag{6.4}$$

Proposition 6.1. *No Love waves, propagating in an orthotropic layer in an orthotropic half-space exist when multiple roots arise in Christoffel's equation for the layer.*

Proof. Expressions (4.5) and (6.1) show that the multiple root γ_1 is necessarily zero. For this case we have

$$C_2 = 0, \quad C_3 = 0, \quad C_1 = 0 \tag{6.5}$$

The first equality follows from the condition for there to be no surface forces (5.8), (5.9) and expression (6.3), the second follows from the condition on the interface (5.10), expressions (6.4) and the first equality of (6.5), while the last equality follows from the second equality of (6.5), which is the condition for there to be no displacements at the interface.

Eliminating the multiple roots, we will consider the case of simple roots. Applying relations (5.2) to the orthotropic layer we obtain

$$\mathbf{M}_1(x') = \left\| \begin{matrix} \operatorname{sh}(ir\gamma_1 x') & \operatorname{ch}(ir\gamma_1 x') \\ ir\gamma_1 G_1 \operatorname{ch}(ir\gamma_1 x') & ir\gamma_1 G_1 \operatorname{sh}(ir\gamma_1 x') \end{matrix} \right\| \tag{6.6}$$

Apart from the scalar factor $ir\gamma_1 G_1$, the vectors defined by the second formula of (5.9) and the second formula of (5.11) can be represented in the form

$$\tilde{\mathbf{T}}_1(h_1) = (-\operatorname{sh}(ir\gamma_1 h_1), \operatorname{ch}(ir\gamma_1 h_1)), \quad \tilde{\mathbf{W}}_2 = (-ir\gamma_2 G_2, 1) \tag{6.7}$$

Substituting expressions (6.6) and (6.7) into Eq. (5.12) we obtain after reduction

$$\omega = \frac{c}{2\gamma_1 h_1}(\operatorname{arctg}(i\zeta_{21}) + n\pi), \quad n = 0, 1, 2, \ldots; \quad \zeta_{21} = \frac{\gamma_2 G_2}{\gamma_1 G_1} \qquad \square \tag{6.8}$$

Proposition 6.2.

A. *In the case considered Love waves can only propagate when the phase velocity lies in the range (c_1^T, c_2^T), where $c_k^T = \sqrt{E_k/\rho_k}(k=1,2)$ are the velocities of the corresponding shear body waves with polarization vector* ***m*** *(for this range all the roots of Christoffel's equation are simple).*
B. *For a fixed frequency ω not more than a finite number of Love waves can exist, which propagate with different phase velocities $c \in (c_1^T, c_2^T)$.*
C. *For a fixed velocity $c \in (c_1^T, c_2^T)$ a denumerable set of Love waves exists, which propagate with different frequencies ω.*

Proof. We will assume that $c_1^T < c_2^T$ (the range of velocities is non-empty). An analysis of expression (6.8) show that when $c \in (c_1^T, c_2^T)$, according to formula (6.1), the Christoffel parameter γ_1 is real and negative, whereas the parameter γ_2 is imaginary with a negative imaginary part. Substituting the expressions for γ_1 and γ_2 into formula (6.8), we obtain positive values of the phase velocity ω. Now assuming that $c < c_1^T$, we obtain

$$\omega = -\frac{c}{2|\gamma_1|h_1}\operatorname{arth}|\zeta_{21}|$$

i.e. $\omega < 0$, which is impossible. Part A is proved. The remaining parts of the assertion follow directly from expression (6.8). □

Corollary 6.1. *No Love waves can exist in the case considered if* $c_1^T > c_2^T$.

6.2. Two orthotropic layers on an orthotropic half-space (simple roots)

In addition to the existing notation we will introduce the following notation

$$\xi_k = 2r\gamma_k h_k, \quad \zeta_{kl} = \frac{\gamma_k G_k}{\gamma_l G_l}, \quad k, l = 1, 2, 3$$

The subscripts 1 and 2 relate to the layers while the subscript 3 relates to the half-space. We obtain the following expression for the product of the transfer matrices

$$\mathbf{M}_2(-h_2)\cdot\mathbf{M}_2^{-1}(h_2)\cdot\mathbf{M}_1(-h_1) = \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\|$$

$$a_{11} = -i\left(\cos\xi_2\sin\frac{\xi_1}{2} + \zeta_{12}\sin\xi_2\cos\frac{\xi_1}{2}\right), \quad a_{12} = \cos\xi_2\cos\frac{\xi_1}{2} - \zeta_{12}\sin\xi_2\sin\frac{\xi_1}{2} \tag{6.9}$$

$$a_{21} = -ir\gamma_2 G_2\left(\sin\xi_2\sin\frac{\xi_1}{2} - \zeta_{12}\cos\xi_2\cos\frac{\xi_1}{2}\right), \; a_{22} = r\gamma_2 G_2\left(\sin\xi_2\cos\frac{\xi_1}{2} + \zeta_{12}\cos\xi_2\sin\frac{\xi_1}{2}\right)$$

The expressions for the vectors $\tilde{\mathbf{T}}_1$ and $\tilde{\mathbf{W}}_3$ remain the same as in (6.7), but with the obvious change in subscripts. Substituting expressions (6.9) into Eq. (5.12) we obtain the resolvent in the form

$$\sin\xi_2(\cos\xi_1 + i\zeta_{32}\zeta_{12}\sin\xi_1) - \cos\xi_2(i\zeta_{32}\cos\xi_1 - \zeta_{12}\sin\xi_1) = 0 \tag{6.10}$$

Unlike the previous case, Eq. (6.10), generally speaking, is unsolvable for the phase frequency ω.

6.3. Two orthotropic half-spaces with an orthotropic layer between them

Bearing Corollary 3.1 in mind, we can conclude from Proposition 3.1 that the case which leads to multiple roots for the half-spaces must be eliminated from consideration. At the same time, corresponding to Remark 6.1.A, multiple roots arise in Christoffel's equation for an orthotropic layer only when the phase velocity is identical with the velocity of the transfer body wave, and the corresponding Christoffel parameter γ_2 becomes zero. Further, as a result of discussions similar to those used when proving Proposition 6.1, we arrive at the following assertion.

Proposition 6.3. *No Love waves exist in the system formed by two orthotropic half-spaces and an orthotropic layer between them if a multiple root appears in Christoffel's equation for the layer. Excluding multiple roots, we will consider the case of simple roots in Christoffel's equation for the layer. The assumption that, in both half-spaces, the wave attenuates with depth, compels us to find the solution for the phase velocities satisfying the inequality*

$$c < \min(c_1^T, c_3^T) \tag{6.11}$$

where c_1^T, c_3^T are the velocities of horizontally polarized transverse body waves, propagating in the corresponding half-spaces in the direction of the vector $\boldsymbol{n}$. *Condition (6.11) ensures that the imaginary parts of the Christoffel parameters* γ_1 *and* γ_3 *are non-zero.*

Remark 6.2. The attenuation with depth in the "upper" half-space (when $x' \to +\infty$) is ensured by the choice of γ_1 with positive imaginary part.

Taking the limit as $h_1 \to \infty$ in Eq. (6.10) and taking Remark 6.2 into account, we obtain the required dispersion relation in the form

$$\omega = \frac{c}{2\gamma_2 h_2}\left(\operatorname{arctg}\left(i\frac{\zeta_{32}-\zeta_{12}}{1-\zeta_{12}\zeta_{32}}\right)+n\pi\right), \quad n = m, m+1, m+2, \dots \tag{6.12}$$

The parameter m ensures that the frequencies ω are positive; it will be determined explicitly below. It is obvious that when $\zeta_{12}=0$ (a vacuum) dispersion relation (6.12) becomes relation (6.8).

Proposition 6.4.

A. *In the system considered a Love wave can only propagate when*

$$c \in (c_2^T, \min(c_1^T, c_3^T)) \tag{6.13}$$

B. *At a fixed frequency ω not more than a finite number of Love waves can exist, which propagate with different velocities in the range (6.13).*

C. *For a fixed phase velocity in the range (6.13) (provided it is non-empty), a denumerable number of Love waves exist, which propagate with different frequencies ω.*

Proof. If $c > \min(c_1^T, c_3^T)$, a Love wave cannot propagate, since the condition for attenuation in the half-spaces is not satisfied. We will assume that $c < \min(c_1^T, c_2^T, c_3^T)$, in which case all the parameters γ_k turn out to be pure imaginary:

$$\gamma_1 = +i|\gamma_1|, \quad \gamma_2 = \pm i|\gamma_2|, \quad \gamma_3 = -i|\gamma_3| \tag{6.14}$$

When choosing the signs, Remark 6.2 is taken into account. We will introduce the following notation

$$\eta_\pm = \frac{|\zeta_{32}|+|\zeta_{12}|}{1 \pm |\zeta_{32}||\zeta_{12}|}$$

Substituting the values (6.14) into expression (6.12) we obtain

$$\omega = -\frac{c}{2|\gamma_2|h_2}\operatorname{arth}\eta_+ \tag{6.15}$$

But it then turns out that $\omega < 0$, which is impossible.

It remains to show that when the phase velocity lies in the range (6.13), the corresponding frequency ω is positive. For this range of parameters, γ_1 and γ_2 satisfy the first and third relations of (6.14) respectively, and $\gamma_2 = \pm|\gamma_2|$. As before, we obtain

$$\omega = -\frac{c}{2|\gamma_2|h_2}(\operatorname{arctg}\eta_- + n\pi), \quad n = m, m+1, m+2, \dots \tag{6.16}$$

The integral parameter m is found from the condition

$$m = -\operatorname{entier}(\pi^{-1}\operatorname{arctg}\eta_-) \tag{6.17}$$

which ensures that ω is positive. Part A is proved.

The proof of parts B and C of the assertion follows directly from relation (6.16). □

Corollary 6.2. *A Love wave cannot propagate if $c_2^T > \min(c_1^T, c_3^T)$.*

Remark 6.3. The results obtained in Section 6 may explain the occurrence of a high-frequency waveguide for waves with transverse horizontal polarization, propagating in a system which is an orthotropic layer between orthotropic half-spaces.

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